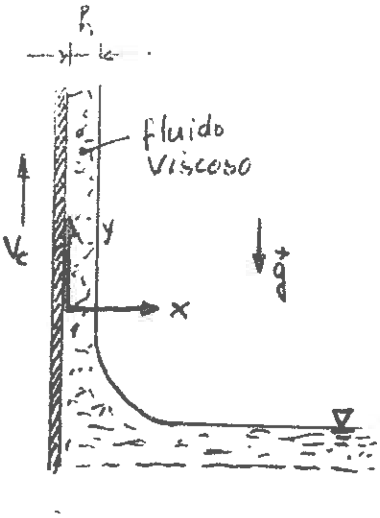
Ex:

Determinar $\bar{v}$ do fluido no escoamento provocado pelo movimento vertical da correia



Assume-se que o escoamento é laminar, estacionário e uniforme

$$\vec{g} = (g_x, g_y, g_z) = (0, -g, 0)$$

$$\vec{V} = (u, v, w) = (0, v, 0)$$

L.c.m. $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \longrightarrow \frac{\partial v}{\partial y} = 0$

$$\frac{\partial v}{\partial t} = 0 \qquad \hookrightarrow v = v(x)$$

L.c.q.d.m. (N-S)

$$p_{sup} \equiv p_{atm} \Rightarrow p = c^{te} \longrightarrow \frac{\partial p}{\partial y} = 0$$

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \longrightarrow \frac{\partial p}{\partial x} = 0$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \longrightarrow -\rho g + \mu\frac{\partial^2 v}{\partial x^2} = 0$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu(\;\ldots\ldots\;) = \rho(\;\ldots\ldots\ldots\;) \longrightarrow \frac{\partial p}{\partial z} = 0$$

$$\frac{d^2 v}{dx^2} = \frac{\rho g}{\mu} \longrightarrow \frac{dv}{dx} = \frac{\rho g}{\mu}x + C_1$$

Assume-se que, para $x = h$: $\tau_{xy} = 0 \Rightarrow \left.\frac{dv}{dx}\right|_{x=0} = 0 \quad \hookrightarrow C_1 = -\frac{\rho g}{\mu}h$

$$\frac{dv}{dx} = \frac{\rho g}{\mu}x - \frac{\rho g}{\mu}h \longrightarrow v = \frac{\rho g}{2\mu}x^2 - \frac{\rho g h}{\mu}x + C_2 \qquad x=0 \therefore v = V_0 \Rightarrow C_2 = V_0$$

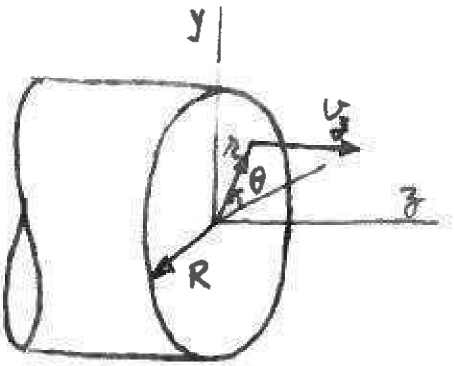
$$\Rightarrow v = V_0 + \frac{\rho g}{2\mu}x^2 - \frac{\rho g h}{\mu}x$$

Caudal: $\quad Q = \int_A v\,dA = \int_0^h v\,B\,dx \longrightarrow \ldots\; Q = B\left(V_0 h - \frac{\rho g h^3}{3\mu}\right)$

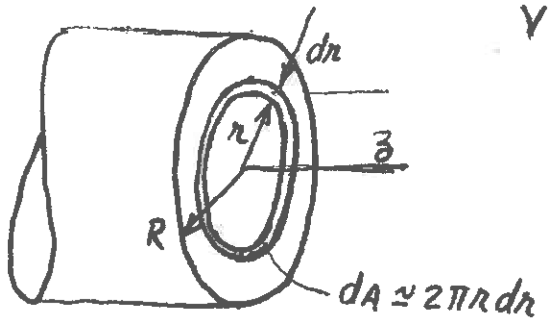
velocidade média: $\quad \bar{v} = \frac{Q}{A} = \frac{Q}{Bh} \longrightarrow \ldots\; \bar{v} = V_0 - \frac{\rho g h^2}{3\mu}$

para $\bar{v} > 0 \Rightarrow V_0 > \rho g h^2 / 3\mu$

Ex: Esc. laminar, estacionário e incompressível num tubo cilíndrico – esc. Hagen-Poiseuille



$\downarrow \vec{g}$



$$\vec{V} = (v_r, v_\theta, v_z) = (0, 0, v_z)$$

$$\vec{g} = (g_r, g_\theta, g_z) = (-g\, \text{sen}\,\theta, -g\cos\theta, 0)$$

• L.C.M.: $\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \longrightarrow \frac{\partial v_z}{\partial z} = 0$

• Eq. N-S: .....
$$\begin{cases} 0 = -\rho g\, \text{sen}\,\theta - \frac{\partial p}{\partial r} \\ 0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \\ 0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial \theta}\right)\right] \end{cases}$$

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$p = -\rho g r\, \text{sen}\,\theta + f_1(z)$ $\qquad v_z = \frac{1}{4\mu}\frac{\partial p}{\partial z}(r^2 - R^2)$

$Q = \int_A v_z\, dA = \dots = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z}$ $\qquad V = \frac{Q}{A} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial z}$

$v_{max} = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}$ $\qquad \hookrightarrow V = \frac{1}{2} v_{max}$

$\frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2$

Nota: $-\frac{\partial p}{\partial z} \equiv \frac{\Delta p}{L}$ queda de p por unidade de comp.

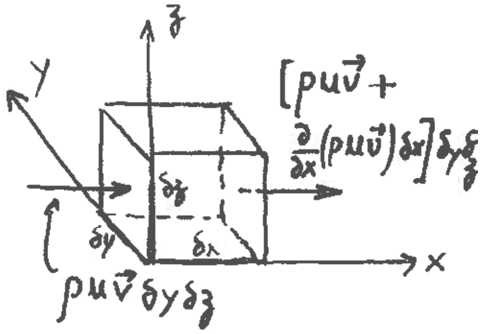
$\hookrightarrow Q = \frac{\pi R^4 \Delta p}{8 \mu L}$ para um dado valor de $\frac{\Delta p}{L}$, se $R \curvearrowright 2R \Rightarrow Q \curvearrowright 16Q$

L.C.Q.D.M.

T.T.R.: $\sum \vec{F}_{ext} = \frac{\partial}{\partial t} \int_{VC} \rho \vec{V} \, d\forall + \oint_{SC} \rho \vec{V} \, \vec{V}.\vec{n} \, dA$

$$\frac{\partial}{\partial t} \int_{VC} \rho \vec{V} \, d\forall \simeq \frac{\partial}{\partial t} (\rho \vec{V}) \, \delta x \, \delta y \, \delta z$$

$$\oint_{SC} \rho \vec{V} \, \vec{V}.\vec{n} \, dA = \left[\sum \rho_i \vec{V}_i \, V_i \, A_i \right]_s - \left[\dots \right]_e$$



face	ent.	saída	balanço
x	$\rho u \vec{V} \delta y \delta z$	$\left[\rho u \vec{V} + \frac{\partial}{\partial x}(\rho u \vec{V}) \delta x\right] \delta y \delta z$	$\frac{\partial}{\partial x}(\rho u \vec{V}) \delta x \delta y \delta z$
y	. . .	. . . .	$\frac{\partial}{\partial y}(\rho v \vec{V}) \delta x \delta y \delta z$
z	. . .	. . . .	$\frac{\partial}{\partial z}(\rho w \vec{V}) \delta x \delta y \delta z$

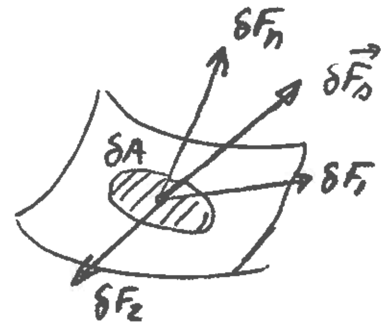
$$\sum \vec{F}_{ext} = \left[\frac{\partial}{\partial t}(\rho \vec{V}) + \frac{\partial}{\partial x}(\rho u \vec{V}) + \frac{\partial}{\partial y}(\rho v \vec{V}) + \frac{\partial}{\partial z}(\rho w \vec{V}) \right] \delta x \, \delta y \, \delta z \quad [N]$$

$$\sum \vec{f}_{ext} = \frac{\partial}{\partial t}(\rho \vec{V}) + \frac{\partial}{\partial x}(\rho u \vec{V}) + \frac{\partial}{\partial y}(\rho v \vec{V}) + \frac{\partial}{\partial z}(\rho w \vec{V}) \quad [N/m^3]$$

$$= \vec{V} \left[\cancel{\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{V})} \right]^{=0} + \rho \underbrace{\left(\frac{\partial \vec{V}}{\partial t} + u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} \right)}_{\rho \frac{d\vec{V}}{dt} = \rho \vec{a}}$$

Forças exteriores:

- $\delta \vec{F}_g = \rho \, d\forall \, \vec{g} \rightarrow \vec{f}_g = \rho \vec{g}$
- Forças superficiais $\delta \vec{F}_s = (\delta F_n, \delta F_1, \delta F_2)$



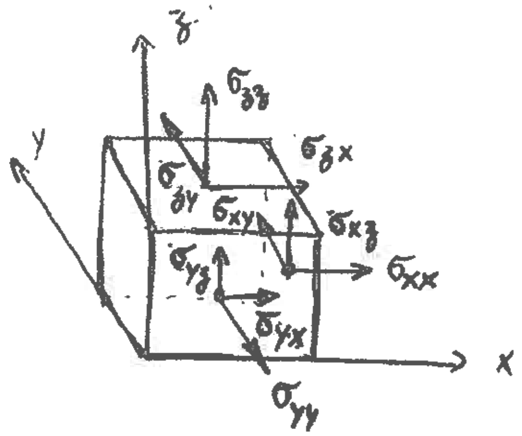
δF_n – normal a δA

$(\delta F_1 \perp \delta F_2)$ – paralelas a δA

Tensão normal: $\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$

Tensões tangenciais: $\sigma_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$ $\quad \sigma_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$

σ_{ij} $\begin{cases} i \rightarrow \text{direcção} \perp \text{ao plano} \\ j \rightarrow \text{direcção da tensão} \end{cases}$



Balanço na direcção x:

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right) \delta x\, \delta y\, \delta z$$

$$f_{sx} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \qquad f_{sy} = \cdots \quad f_{sz} = \cdots$$

Equação do movimento (2ª L.N.): $\sum \vec{f}_{ext} = \rho \vec{a} = \rho \frac{d\vec{v}}{dt}$

$$\begin{cases} f_{gx} + f_{sx} = \rho a_x \\ f_{gy} + f_{sy} = \rho a_y \\ f_{gz} + f_{sz} = \rho a_z \end{cases} \qquad \begin{cases} a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y = \cdots \\ a_z = \cdots \end{cases}$$

$$\begin{cases} \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \\ \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \\ \rho g_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \end{cases}$$

- As forças superficiais resultam da pressão hidrostática, p, e da tensão viscosa τ_{ij} associada a gradientes de velocidade (L. de N. da V.)

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

resultando:

$$\begin{cases} \rho g_x - \dfrac{\partial p}{\partial x} + \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) \\ \rho g_y - \dfrac{\partial p}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) \\ \rho g_z - \dfrac{\partial p}{\partial z} + \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \tau_{zz}}{\partial z} = \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) \end{cases}$$

ou $\boxed{\rho\vec{g} - \vec{\nabla}p + \vec{\nabla}\cdot\vec{\tau}_{ij} = \rho\dfrac{d\vec{v}}{dt}}$ com $\vec{\tau}_{ij} = \tau_x\vec{i} + \tau_y\vec{j} + \tau_z\vec{k}$

↳ Equações diferenciais gerais do movimento de um fluido

- Escoamento invíscido $\rightarrow \tau_{ij} = 0$

↳ $\boxed{\rho\vec{g} - \vec{\nabla}p = \rho\dfrac{d\vec{v}}{dt}}$ (eq. de Euler)

↳ equações mais simples, mas ainda de resolução complexa devido, principalmente aos termos não-lineares ($u\,\partial u/\partial x$, $v\,\partial u/\partial y$, ...)

- Fluido newtoniano e incompressível

são válidas as relações:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

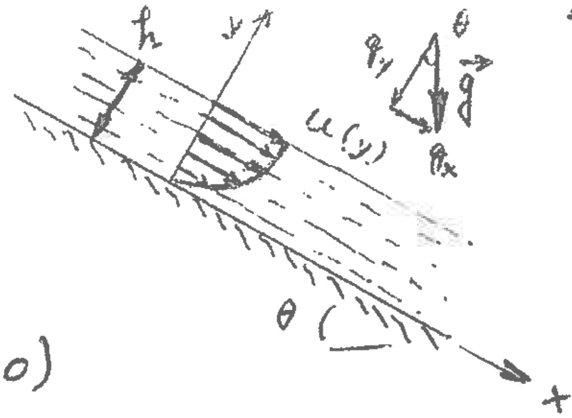
$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$\Sigma f_x = \rho a_x$

$$\begin{cases} \rho g_x - \dfrac{\partial p}{\partial x} + \mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) \\ \rho g_y - \dfrac{\partial p}{\partial y} + \mu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) = \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) \\ \rho g_z - \dfrac{\partial p}{\partial z} + \mu\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) = \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) \end{cases}$$

4 incog. (u, v, w, p) ∴ 4 eq. (3 qdm + 1 c.m.)

→ EQUAÇÕES DE NAVIER-STOKES: $\rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{V} = \rho\dfrac{d\vec{v}}{dt}$



Esc. estacionário, laminar e incompressível.

$$\vec{V} = (u, v, w) = (u, 0, 0)$$

$$\vec{g} = (g_x, g_y, g_z) = (g \operatorname{sen}\theta, -g\cos\theta, 0)$$

L.C.M.: $\vec{\nabla}\cdot\vec{V} = 0 \longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \longrightarrow \frac{\partial u}{\partial x} = 0 \longrightarrow u = u(y)$

N.S.: $\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$$\begin{cases} \rho(0) = \rho g \operatorname{sen}\theta + \mu \frac{\partial^2 u}{\partial y^2} \\ \rho(0) = -\rho g\cos\theta - \frac{\partial p}{\partial y} \longrightarrow p = -\rho g \cos\theta\, y + \overbrace{f(x,y)}^{=0} \\ \rho(0) = -\frac{\partial p}{\partial z} \longrightarrow \frac{\partial p}{\partial z} = 0 \end{cases}$$

$$\frac{d^2 u}{dy^2} = -\frac{\rho g \operatorname{sen}\theta}{\mu} \longrightarrow \frac{du}{dy} = -\frac{\rho g \operatorname{sen}\theta\, y}{\mu} + C_1$$

$y = h \rightarrow \tau = 0$

$\tau = \mu \frac{du}{dy} \rightarrow \frac{du}{dy} = 0$

$\hookrightarrow C_1 = \frac{\rho g \operatorname{sen}\theta}{\mu}$

$$u = \frac{\rho g \operatorname{sen}\theta}{\mu}\left(-\frac{y^2}{2} + h y\right) + C_2$$

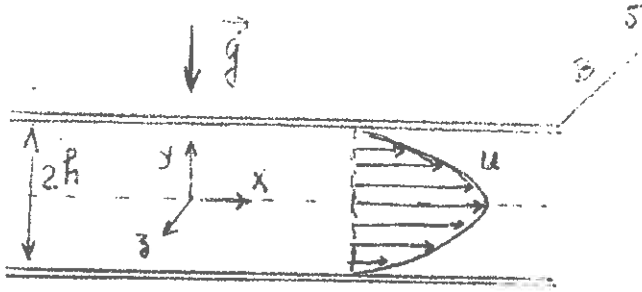
$y = 0 \rightarrow u = 0$

$\hookrightarrow C_2 = 0$

$$\underbrace{u = \frac{\rho g \operatorname{sen}\theta\, y}{2\mu}(2h - y)}$$

$$Q = \int_A u\, dA = \int_0^h B\, u\, dy = \ldots\ldots$$

Ex: Esc. estacionário, laminar e incompressível entre placas planas e paralelas.



$\vec{V} = (u, v, w) = (u, 0, 0) \qquad \vec{g} = (g_x, g_y, g_z) = (0, -g, 0)$

L.C.M.: $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{V}) = 0 \quad \ldots \quad \vec{\nabla}\cdot\vec{V} = 0 \rightarrow \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

$\hookrightarrow \underline{\dfrac{\partial u}{\partial x} = 0} \Longrightarrow \underline{u = u(y)}$

N-S: $\ldots \begin{cases} -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{c)} \quad \dfrac{d^2 u}{dy^2} = \dfrac{1}{\mu}\dfrac{\partial p}{\partial x} \rightarrow \text{n depende de } y \\ -\rho g - \dfrac{\partial p}{\partial y} = 0 & \text{b)} \quad p = -\rho g y + f_1(x) \\ -\dfrac{\partial p}{\partial z} = 0 & \text{a)} \quad p \text{ não depende de } z \end{cases}$

$\dfrac{d^2 u}{dy^2} = \dfrac{1}{\mu}\dfrac{\partial p}{\partial x} \longrightarrow \dfrac{du}{dy} = \dfrac{1}{\mu}\dfrac{\partial p}{\partial x} y + C_1 \longrightarrow u = \dfrac{1}{2\mu}\dfrac{\partial p}{\partial x} y^2 + C_1 y + C_2$

Cond.-F.: $y = \pm h \rightarrow u = 0 \Rightarrow C_1 = 0$ e $C_2 = -\dfrac{1}{2\mu}\dfrac{\partial p}{\partial x} h^2$

$\hookrightarrow \underline{u = \dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}\left(y^2 - h^2\right)}$ L. de distribuição ou perfil de velocidades (parabólico)

Veloc. máx.: $\dfrac{du}{dy} = 0 \longrightarrow \ldots\ y = 0 \Longrightarrow \underline{u_{max} = -\dfrac{h^2}{2\mu}\dfrac{\partial p}{\partial x}}$

Veloc. média: $\bar{V} = \dfrac{1}{A}\displaystyle\int_A u\,dA = \dfrac{1}{2hB}\int_{-h}^{h} \dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}\left(y^2 - h^2\right) B\,dy$

$\hookrightarrow \underline{\bar{V} = \ldots = -\dfrac{h^2}{3\mu}\dfrac{\partial p}{\partial x}} \qquad \hookrightarrow \underline{\dfrac{\bar{V}}{u_{max}} = \dfrac{2}{3}}$

Caudal: $Q = \bar{V} A = \bar{V} \cdot 2hB = \ldots \qquad \dot{M} = \rho \dot{V}$

• Campo de pressões $p(x, y)$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \quad \ldots \quad \frac{\partial p}{\partial x} \text{ também não depende de } x$$

$$\rightarrow \frac{\partial p}{\partial x} = C_3 \qquad p = -\rho g y + f_1(x)$$

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = C_3\, dx - \rho g\, dy$$

$$\int_{p_0}^{p} dp = p - p_0 = \int_0^x C_3\, dx - \rho g \int_0^y dy \rightarrow p = p_0 + \frac{\partial p}{\partial x} x - \rho g y$$

• Tensões τ_{ij}

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu \frac{\partial u}{\partial y} = \cdots = \frac{\partial p}{\partial x} y$$

na parede: $y = h \hookrightarrow \tau_p = \frac{\partial p}{\partial x} h$